Memory Effect for Gravitational Waves

I. Velocity vs Displacement Effect for Gravitational Waves

Lectures given by P. Horvathy

Institut Denis-Poisson CNRS/UMR 7013 - Université de Tours - Université d'Orléans Parc de Grammont, 37200; Tours, FRANCE

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based on joint work P-M Zhang, C. Duval,

Z. K. Silagadze, Q.-L. Zhao, M. Elbistan, J. Balog

Abstract: The motion of particles hit by a burst of gravitation waves is studied. With a Gaussian seed solution and its derivatives only numerical solutions are obtained, however approximations of the wave profile by the (derived) Pöschl-Teller or by the Scarf potential allow us to find analytic solutions. The Displacement Memory Effect proposed by Zel'dovich and Polnarev is recovered when the wave parameters take certain "magic" values.

Road map:

- Memory Effect

  velocity VM Ehlers-Kundt, Braginsky-Thorne . . . p.3

  displacement DM (Zel'dovich-Polnarev) p.4

- Sandwich waves p.5
- Geodesics in Brinkmann coordinates p.6.
- Gaussian profile p.8
- Pöschl-Teller potential p.10
- GW in D = 3+1 dim p.13
- DM / VM for flyby p.14
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- Analytical solution for dPT p.18
- Scarf approximation p.19
- Hodograph p.23
- Conclusion p.26

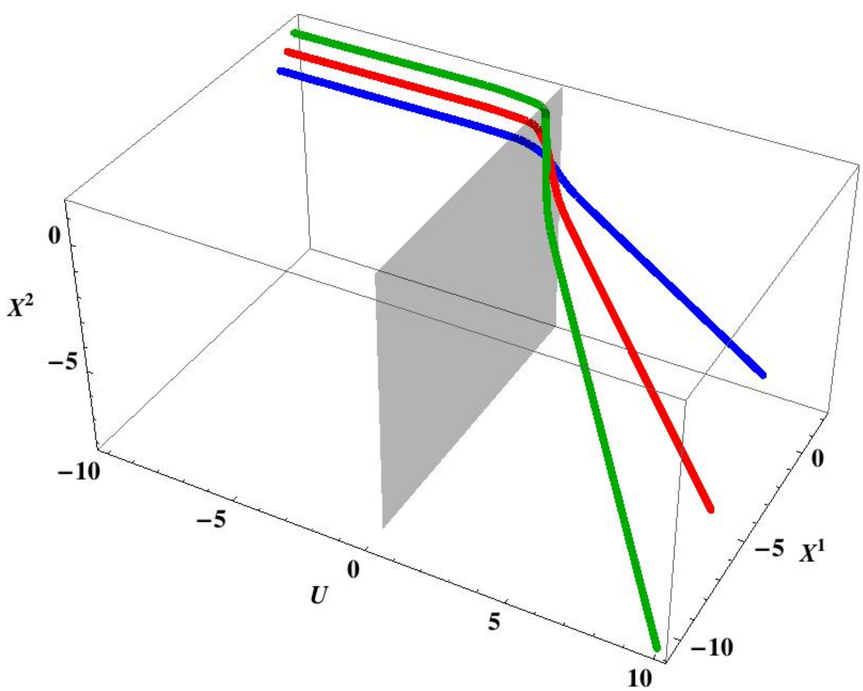
Memory effect

Velocity Effect VM

J. Ehlers and W. Kundt *"Exact solutions of the gravitational field equations," 1962*,

Braginsky and Thorne, *"Gravitational-wave burst with memory and experimental prospects," Nature (London)* **327** *123 (1987).*

Grishchuk and Polnarev, *"Gravitational wave pulses with 'velocity coded memory'," Sov. Phys. JETP* **69** *(1989) 653 [Zh. Eksp. Teor. Fiz.* **96** *(1989) 1153].*



Particles hit by GW fly apart with non-zero constant velocity. "billiard"

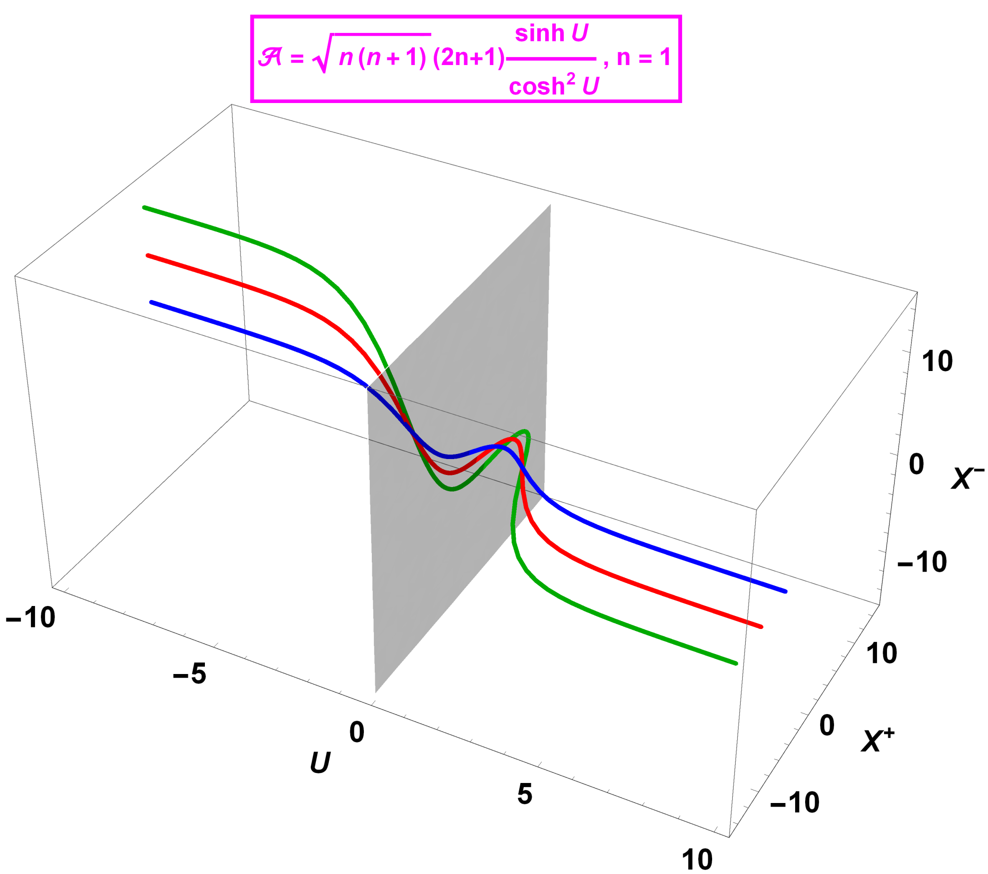
Displacement Effect DM

Zel'dovich, Polnarev *"Radiation of gravitational waves by a cluster of superdense stars," Astron. Zh.* **51**, *30 (1974)*

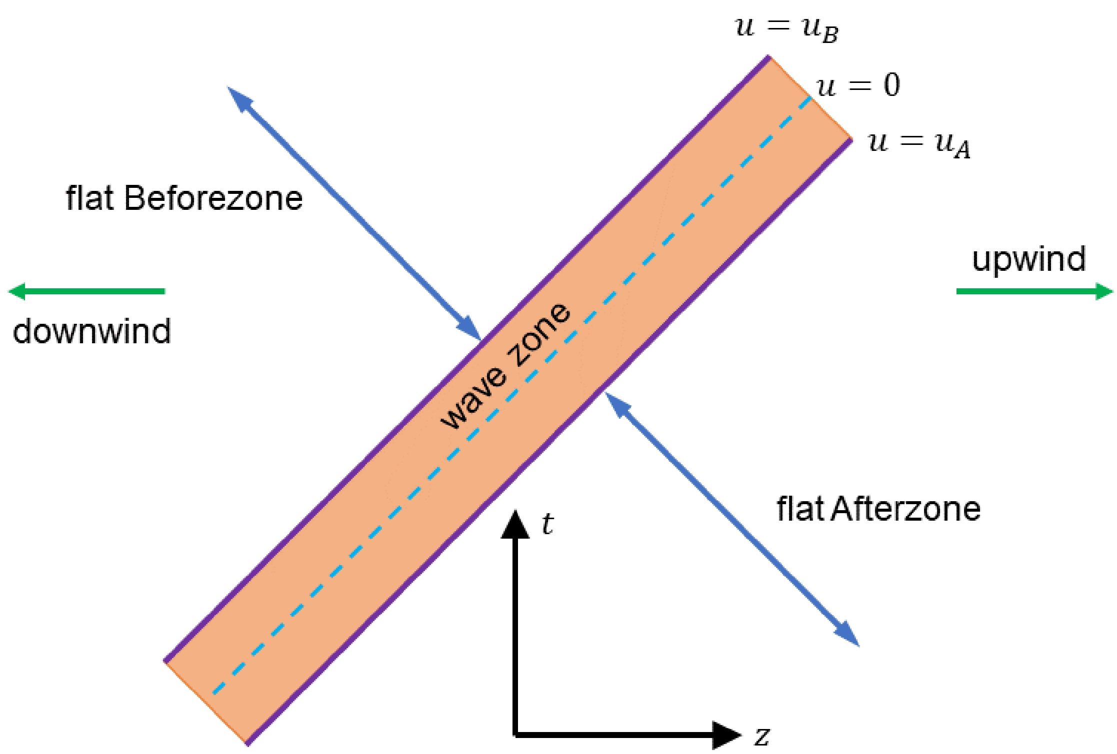
> *...[for] two noninteracting bodies [...] the distance should change, and this effect might possibly serve as a nonresonance detector. [...] their relative velocity will become vanishingly small as* **flyby** *concludes.*

Braginsky & Grishchuk *"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz.* **89** *744-750 (1985)*

Zhang et al **DM** for "magic" wave amplitudes:



Sandwich wave: Spacetime non-flat only in short interval $u_B \leq u \leq u_A$ of retarded time [Wave-zone]. Flat both in Beforezone $u < u_B$ that the wave has not reached yet, and in Afterzone $u_A < u$ where has already passed,



$\big(u = z - t$ flows from left to right, whereas wave advances from right to left.$\big)$

Geodesics in Brinkmann* coordinates

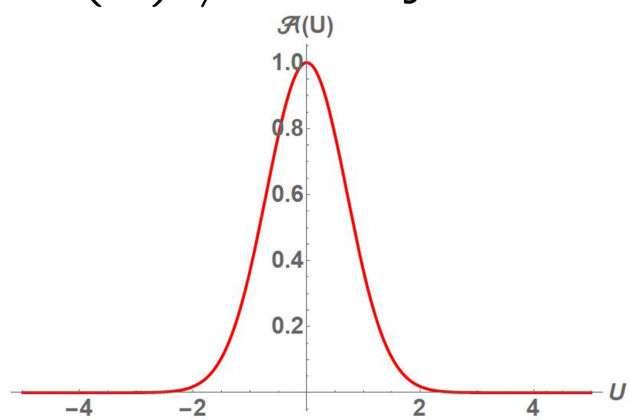
(toy model in $D = 1$ space + 2 lighlike dimensions.) plane GW metric

$$dX^2 + 2dUdV - \mathcal{A}(U)X^2dU^2 \tag{1}$$

X = transverse, U, V light-cone coords.

Sandwich wave: $\mathcal{A}(U) \neq 0$ only in "wave zone"

$U_B < U < U_A$.



For non-tachyonic geodesic: Jacobi invariant

$$\mathfrak{m}^2 = g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = \text{const} \leq 0\,. \tag{2}$$

Massive: $\mathfrak{m}^2 < 0$, Lightlike $\mathfrak{m}^2 = 0$.

* M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

Lightlike geodesics $\mathfrak{m}^2 = 0$:

$$\frac{d^2X}{dU^2} + \frac{1}{2}\mathcal{A}(U)X = 0\,, \tag{3a}$$

$$\frac{d^2V}{dU^2} - \frac{1}{4}\frac{d\mathcal{A}}{dU}(X)^2 - \tfrac{1}{2}\mathcal{A}\,\frac{d(X^2)}{dU} = 0\,. \tag{3b}$$

$V(U)$ horizontal lift of $X(U)$ Coordinate X decoupled from V. Projection into transverse space is V-independent. Conversely, lightlike geo determined by eqn. (3a) with U viewed as Newtonian time *.

Eqn (3a) Sturm-Liouville $\Rightarrow$ no analytic solution in general.

*L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928).

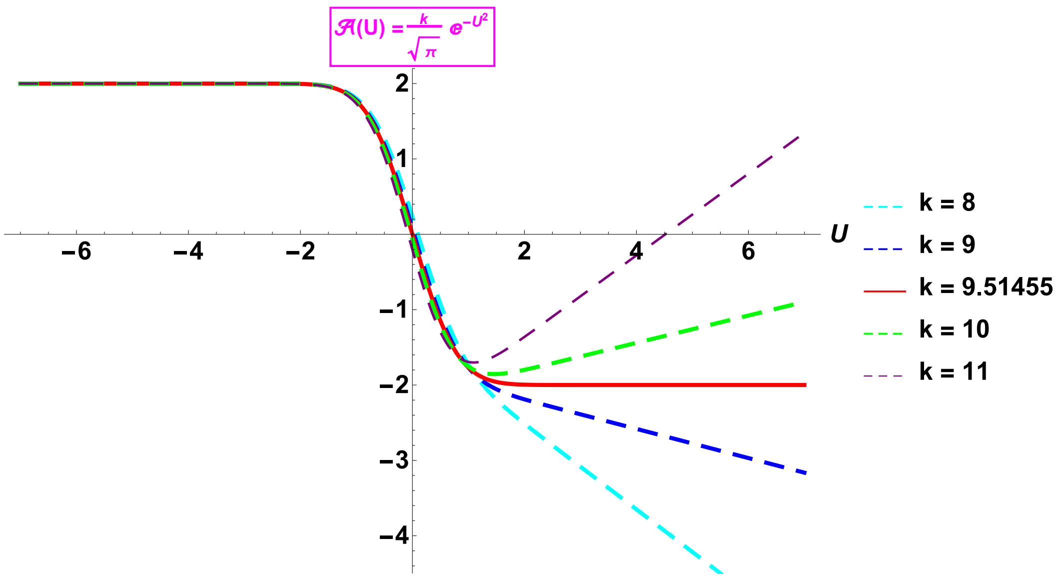
C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-cartan Theory,"* Phys. Rev. D **31** (1985), 1841-1853

C. Duval, G. W. Gibbons and P. Horvathy, *"Celestial mechanics, conformal structures and gravitational waves,"* Phys. Rev. D **43** (1991), 3907-3922 [arXiv:hep-th/0512188 [hep-th]].

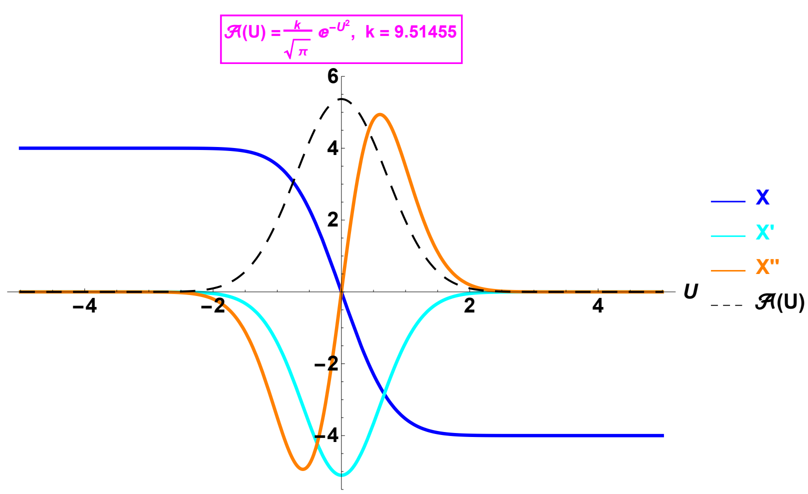
Gaussian profile

$$\mathcal{A}^G(U) = \frac{k}{\sqrt{\pi}} e^{-U^2}, \qquad (4)$$

For general amplitude k particles fly apart with non-zero velocity : VM .

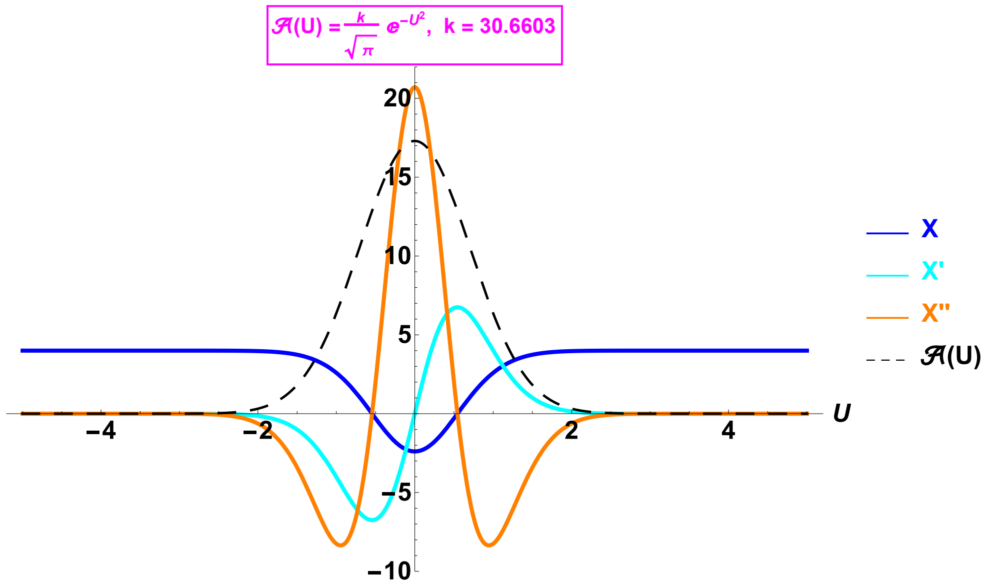


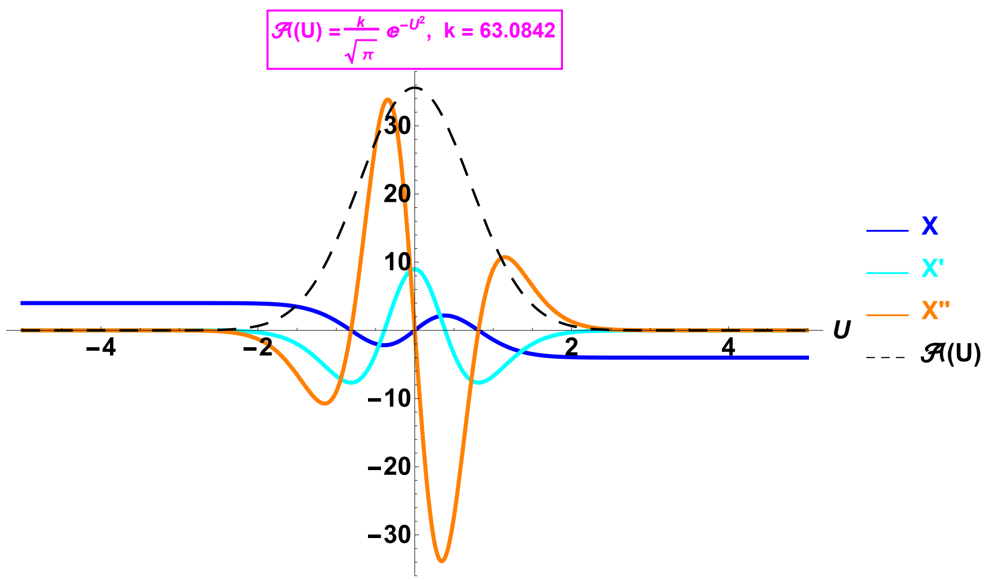
Surprise: fine-tuning amplitude, $k = k_{crit}$ $\Rightarrow$ **DM**



X : **trajectory** , dX/dU : **velocity** , $\mathbf{d^2X}/dU^2$: **force**. $\mathbf{m = 1}$
half-wave.

More fine-tuning $\rightsquigarrow$ higher "magic amplitudes":





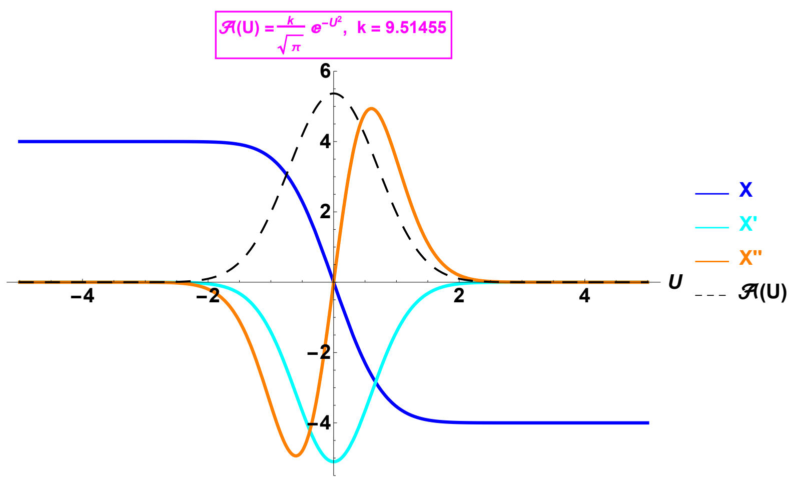
DM *with* $\mathbf{m = 2}$ *and* $\mathbf{m = 3}$ **half-waves** *as trajectories.*

reminiscent of "quantization condition" in Old QM (Balmer, Planck, Bohr, Sommerfeld, etc) - see later.

"Miracle" explained : at (approximate) boundaries of Wavezone $U_B < U < U_A$ both

velocity & force vanish

Outside Wavezone motion governed by Newton's 1st law.



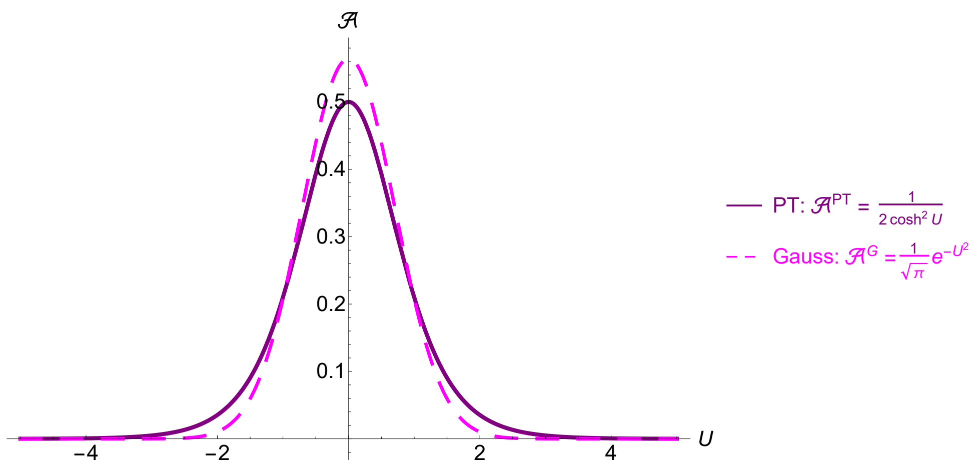
DM trajectories consist of $\mathbf{m}$ half-waves. Outgoing position depends on parity:

$$X_{out} = (-1)^m X_{in} \,. \tag{5}$$

Pöschl-Teller profile

No analytic solutions for Gauss. Shape of $\mathcal{A}^G$ reminiscent of Pöschl-Teller (PT) potential,

$$\mathcal{A}^{PT}(U) = \frac{k}{2\cosh^2 U}\,, \tag{6}$$



Gaussian bell (dashed) well approximated by Pöschl-Teller potential (6) *(solid line), for which* (3a) *has analytic solutions.*

$$k_n = 4n(n+1) \tag{7}$$

$$\boxed{\frac{d^2X}{dU^2} + \frac{n\,(n+1)}{\cosh^2 U} X = 0\,.} \tag{8}$$

$\psi \rightsquigarrow X, \quad x \rightsquigarrow U$ time-indept Schrödinger eqn with $E = 0$ energy. (cf. pp.28-32).

Particle at rest before burst arrives:

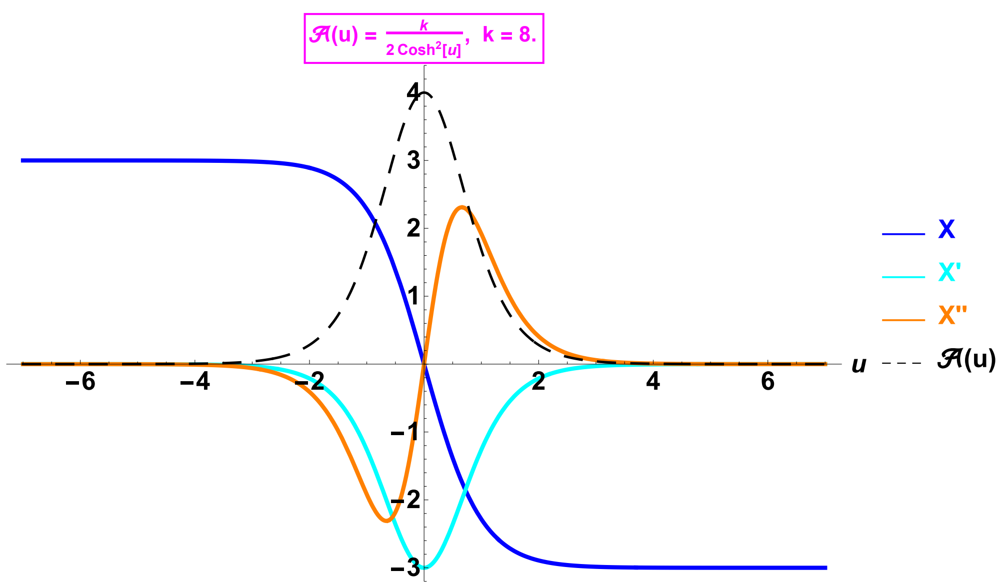
$$X(U=-\infty)=X_0, \quad \dot{X}(U=-\infty)=0\,. \qquad (9)$$

Put $t=\tanh(U)$ into (8) $\rightsquigarrow$ Legendre eqn,

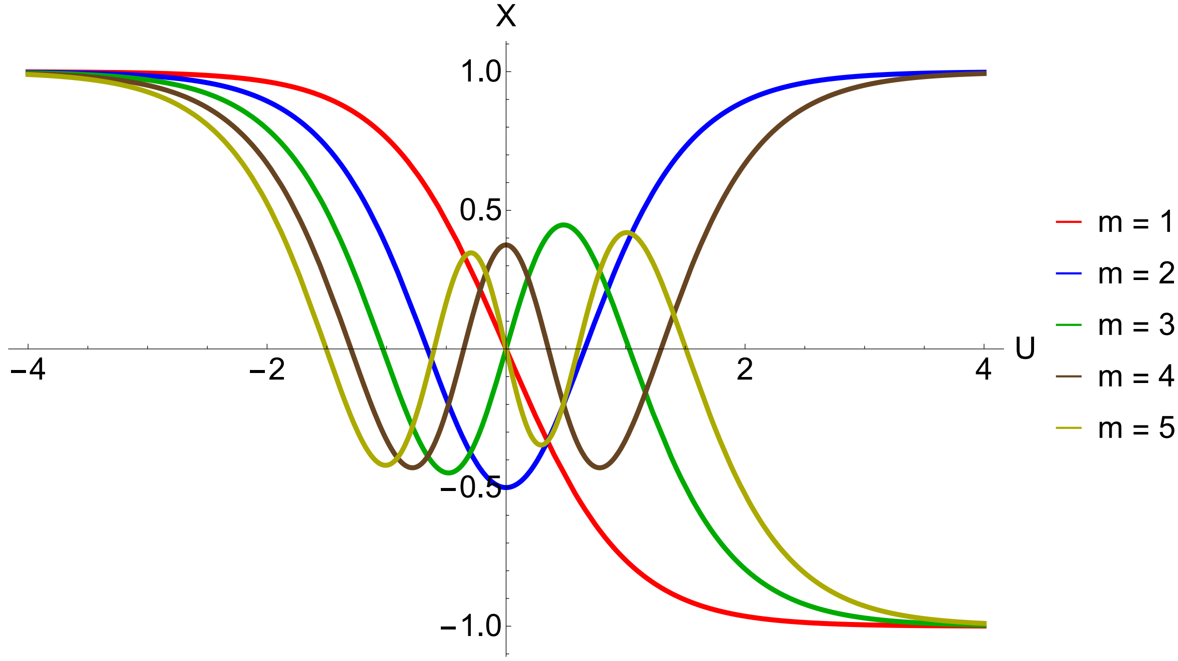
$$\left(1-t^2\right)\frac{d^2X}{dt^2}-2t\frac{dX}{dt}+n\,(n+1)\,X=0\,. \qquad (10)$$

DM requires $X(U)\to$ const for $U\to\infty$ $\Rightarrow$ solution extends to $t=\pm 1$ $\rightsquigarrow$ n positive integer $\Rightarrow$ solution propto Legendre polynomial,

$$X_n(U)=P_n(\tanh U), \quad n=1,\,2,\,\ldots\,, \qquad (11)$$

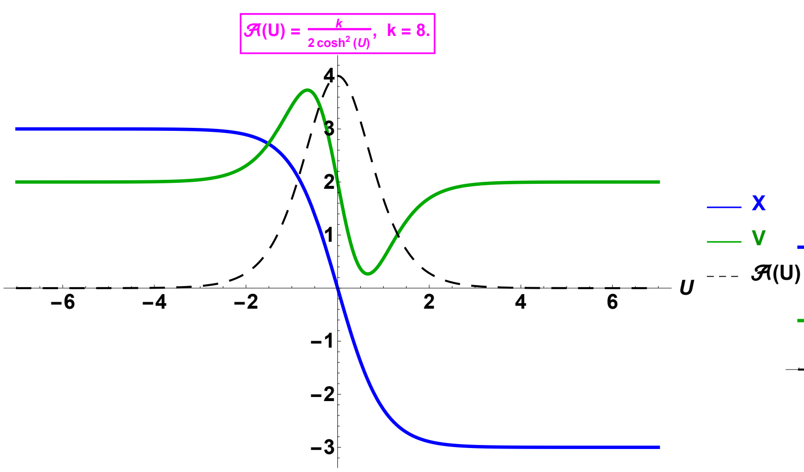


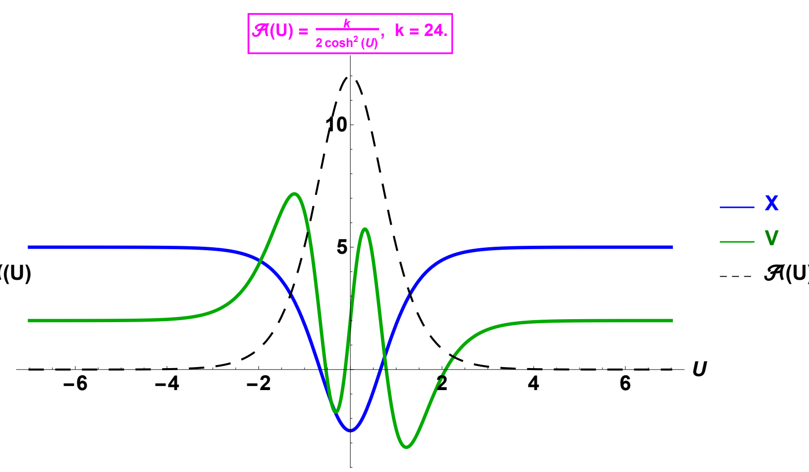
For Pöschl-Teller with $k_1=8$, transverse trajectory consistent with **DM** *for* $\mathbf{n=1}$.



DM *trajectories for Pöschl-Teller profile with* $\mathbf{n = 1, \ldots, 5}$ *half-waves.* $X_{out} = (-1)^n X_{in} \sim$ *time-inversion symmetry*

N.B. "vertical" component $\sim$ minus Hamiltonian action calculated along transverse trajectory





For Pöschl-Teller with k_{crit} *both* **transverse,** X, *and the* **vertical,** V, *trajectories exhibit* **DM** .

GW in $3+1$ dimensions

Coords X^1, X^2, U, V, with U, V lightlike. Metric :

$$\delta_{ij}\, dX^i dX^j + 2dU dV + K_{ij}(U) X^i X^j\, dU^2, \quad (12a)$$

$$K_{ij}(U) X^i X^j = \tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_\times(U)\, X^1 X^2, \quad (12b)$$

Lightlike Geodesics for linearly polarized ($\mathcal{A}_\times \equiv 0$) GW :

$$\frac{d^2X^+}{dU^2} - \frac{1}{2}\mathcal{A}(U) X^+ = 0, \quad (13a)$$

$$\frac{d^2X^-}{dU^2} + \frac{1}{2}\mathcal{A}(U) X^- = 0, \quad (13b)$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}}{dU}\Big((X^+)^2 - (X^-)^2\Big) + \mathcal{A}\Big(X^+\frac{dX^+}{dU} - X^-\frac{dX^-}{dU}\Big) = 0\,. \quad (13c)$$

$X^{1,2}$-components decoupled from V. Projection of $4D$ worldline to transverse plane (X^1, X^2) independent of $V(U)$. Eqn (13c) $\sim$ horizontal lift, follows from (13a)-(13b).

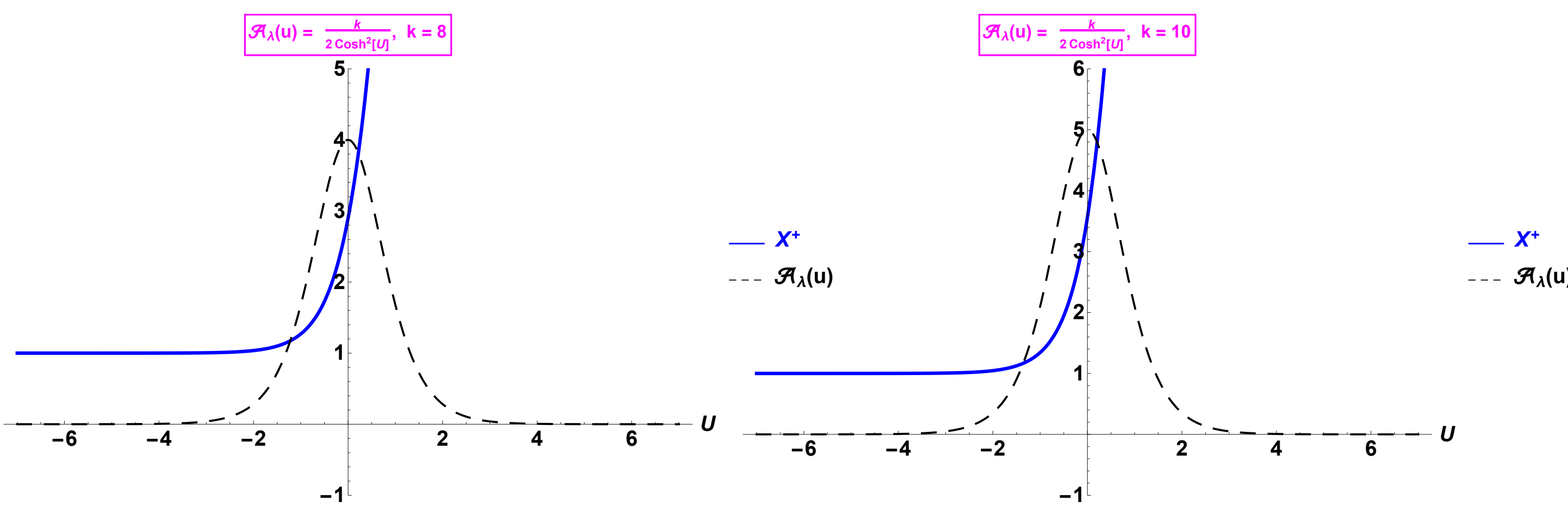
In Eisenhart-Duval framework: (13a)-(13b) $\sim$ repulsive/attractive harmonic force with (time-dept) frequency

$$\boxed{\omega^2(U) = \mp\mathcal{A}(U)} \qquad (14)$$

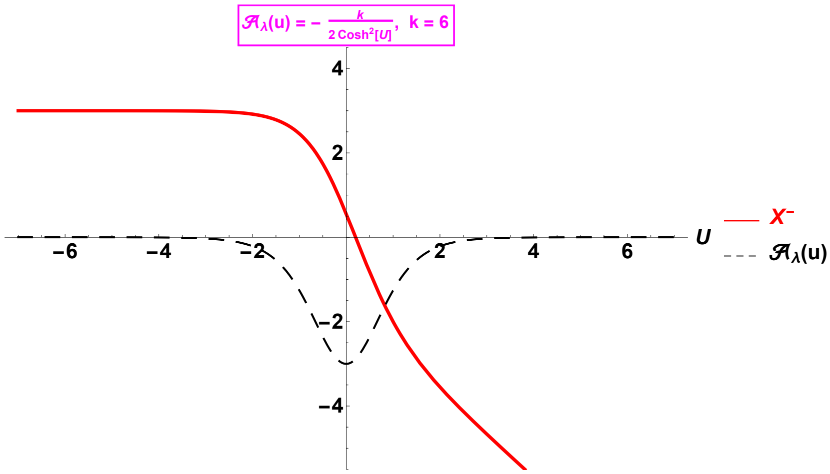
where $\mp \sim$ lower/upper component $X^{\mp} \rightsquigarrow$ Sturm-Liouville.

X^- attractive $\omega^2 > 0 \rightsquigarrow$ DM for magic amplitude.

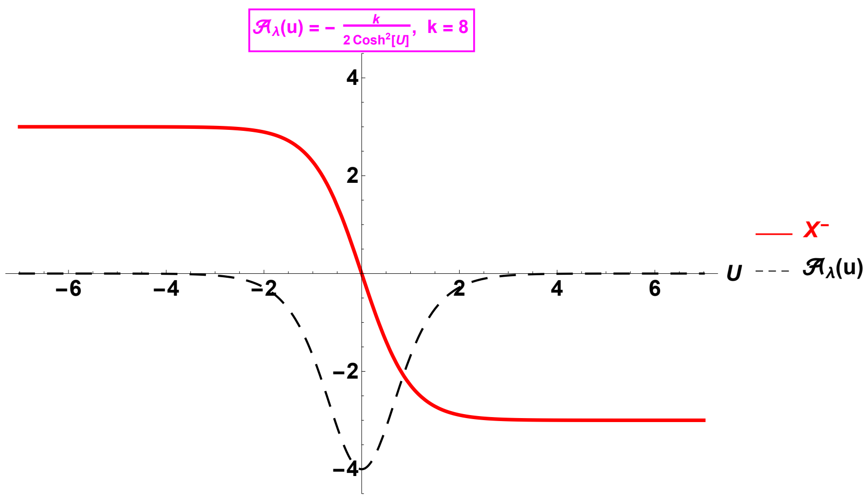
Ex: PT Geodesics in repulsive sector $\omega^2 < 0$



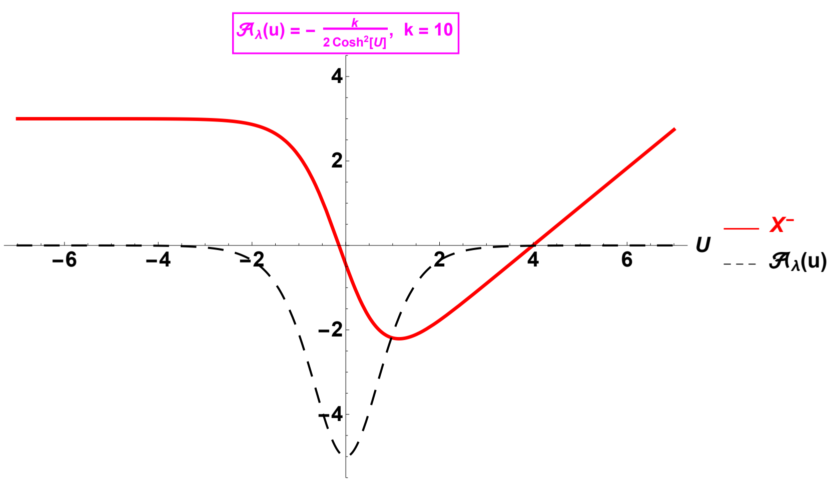
Geodesics in repulsive sector diverge: *NO DM*



force too weak



DM



force too strong

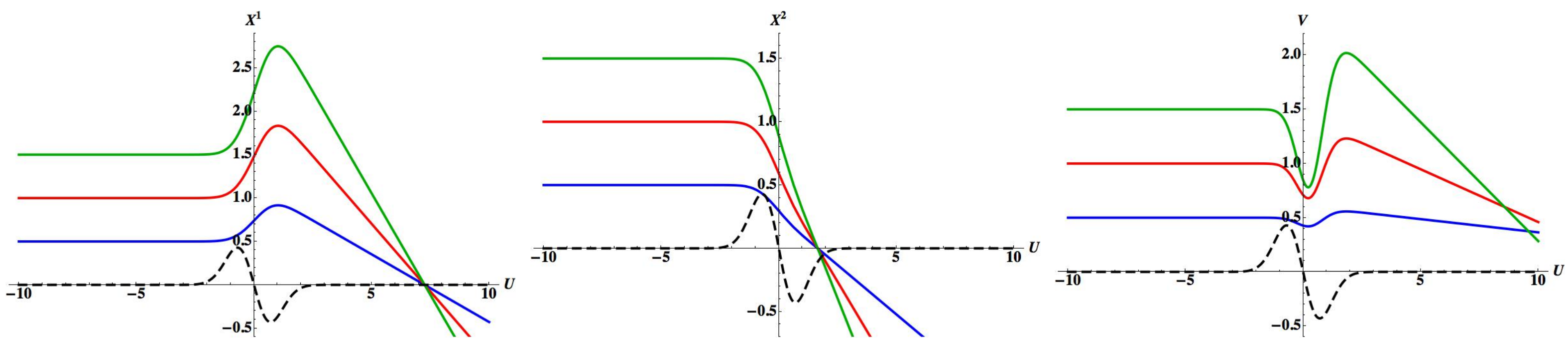
Geodesics in the attractive sector

DM for flyby Zel'dovich-Polnarev

Toy model $D = 2$ * : Profile for **flyby** proportional to *1st derivative* of Gaussian,

$$\mathcal{A}(U) = \frac{1}{2}\frac{d(e^{-U^2})}{dU}\,. \qquad (15)$$

No analytic solution. Numerically found geodesics :

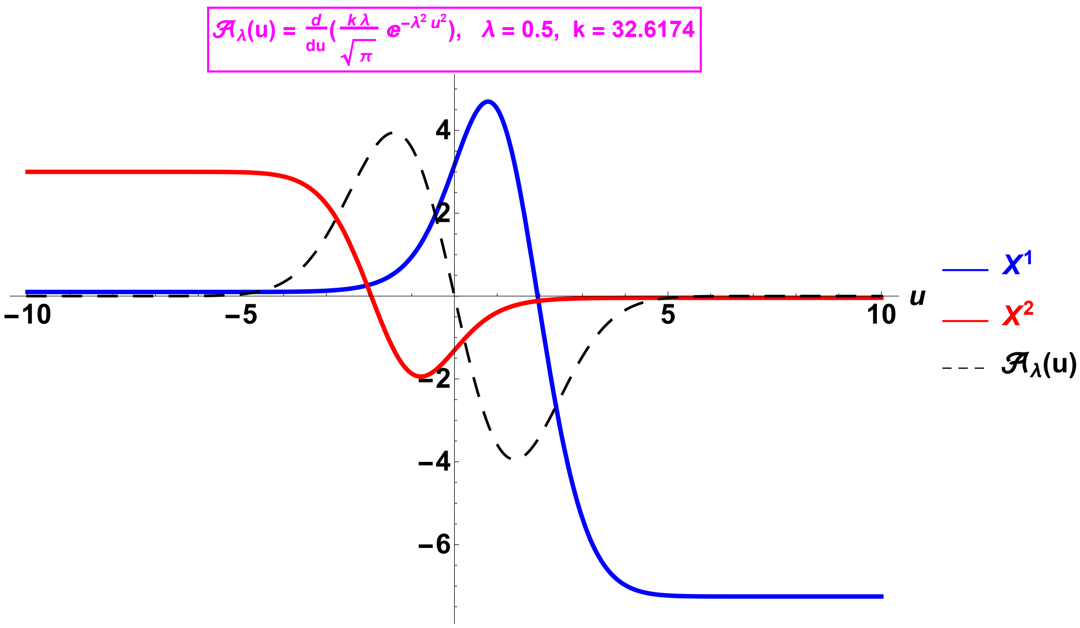


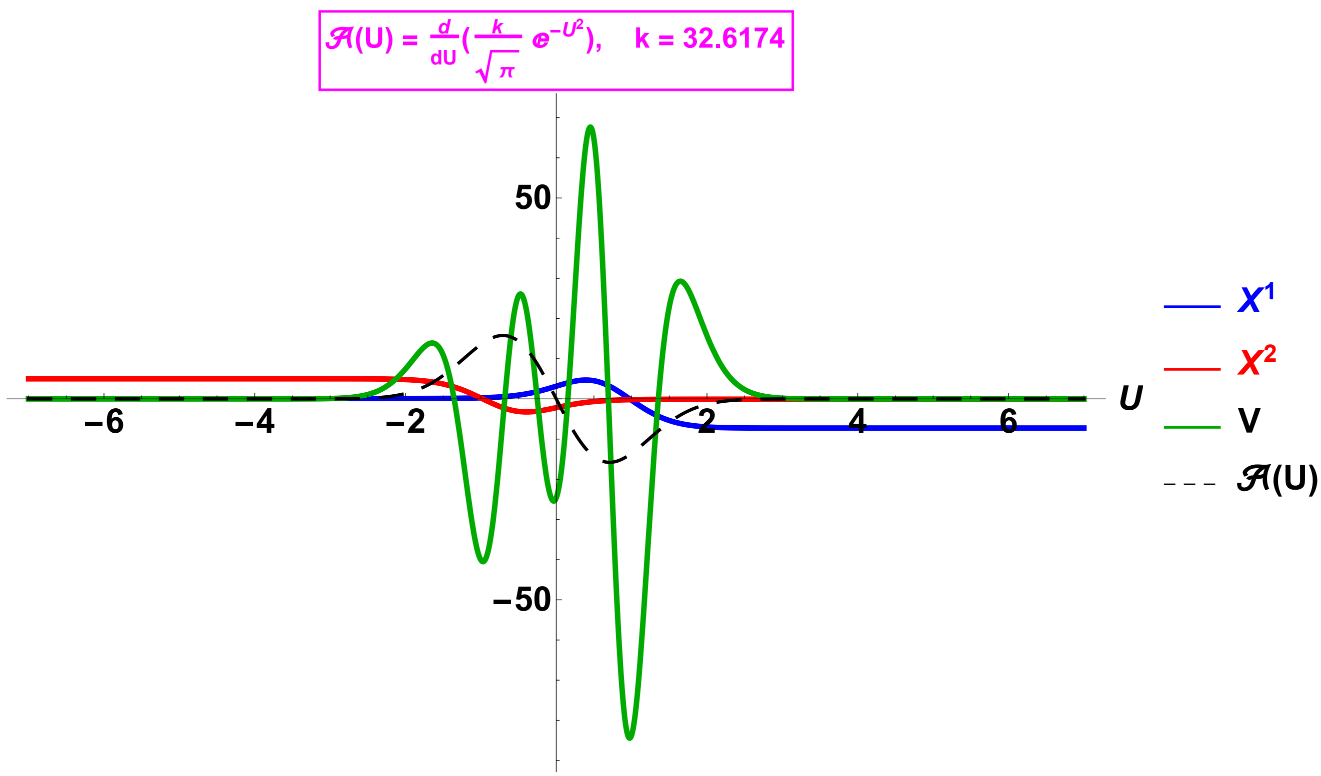
Geodesics for flyby profile (15) with amplitude $k = 1$.

Can VM become **DM** ???

*P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, *"Soft gravitons and the memory effect for plane gravitational waves,"* Phys. Rev. D **96** (2017) no.6, 064013[arXiv:1705.01378 [gr-qc]].

Miracle ! for (numerically found) "magic" parameters $\rightsquigarrow$ DM for both components !





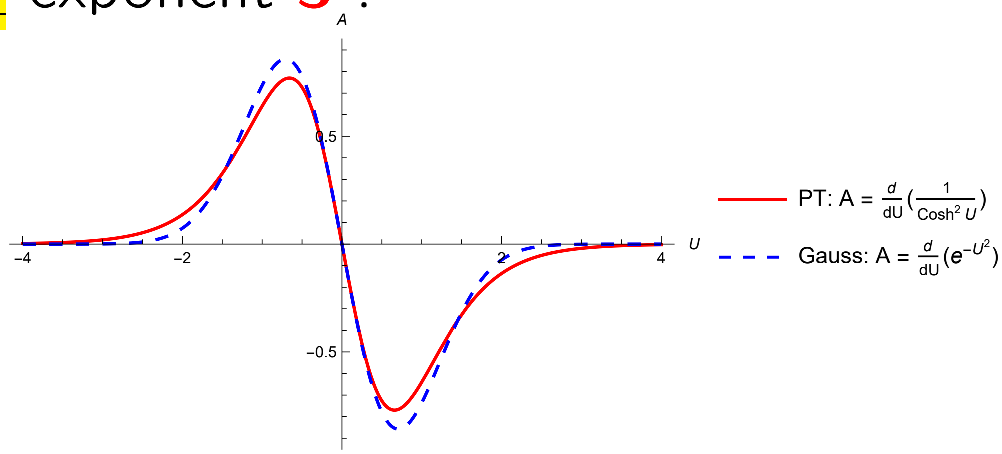
DM for both components ! Reminiscent of:

Derived Pöschl-Teller (dPT) profile

$$K_{ij}(U)X^iX^j = \mathcal{A}^{dPT}\Big((X^1)^2-(X^2)^2\Big), \qquad (16)$$

$$\mathcal{A}^{dPT}(U) = \frac{d}{dU}\left(\frac{g}{2\cosh^2 U}\right) = -g\frac{\sinh U}{\cosh^{\mathbf{3}} U}\,. \qquad (17)$$

N.B. exponent **3** !



Derivative of Pöschl-Teller potential is good approximation of d-Gaussian, (15).

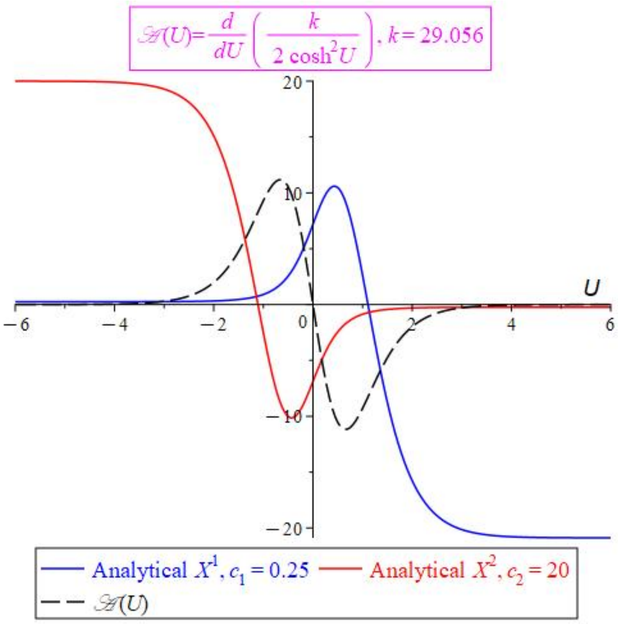
Approximate semi-analytical solution*

For dPT profile (17), putting $t = \tanh U$ eqns (13a)-(13b) become cf. (10)

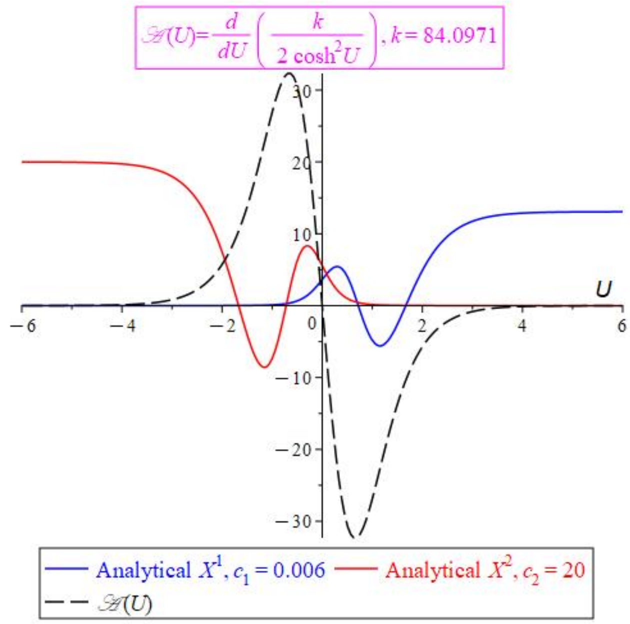
$$(1-t^2)\frac{d^2X^\pm}{dt^2} - 2t\frac{dX^\pm}{dt} \pm \frac{k}{2}tX^\pm = 0\,, \qquad (18)$$

where $\pm$ refers to upper/lower components.

*P. M. Zhang, Q. L. Zhao, J. Balog and P. A. Horvathy, *"Displacement memory for flyby,"* Annals Phys. **473** (2025), 169890 [arXiv:2407.10787 [gr-qc]].

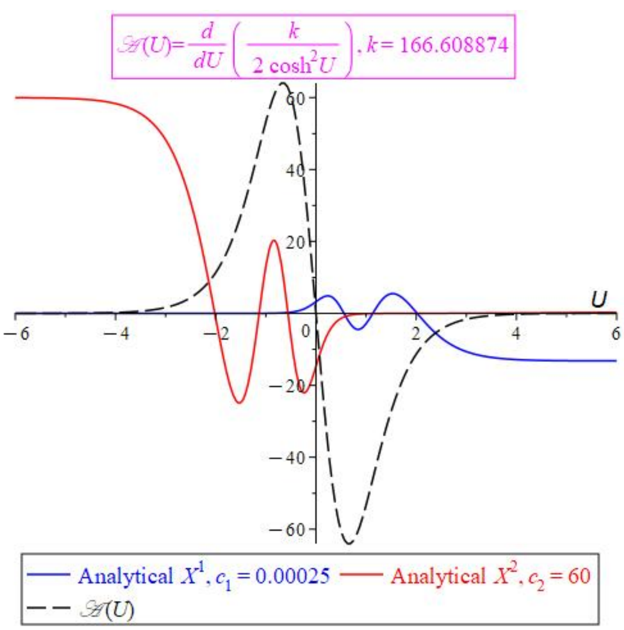


$\mathbf{m = 1}$

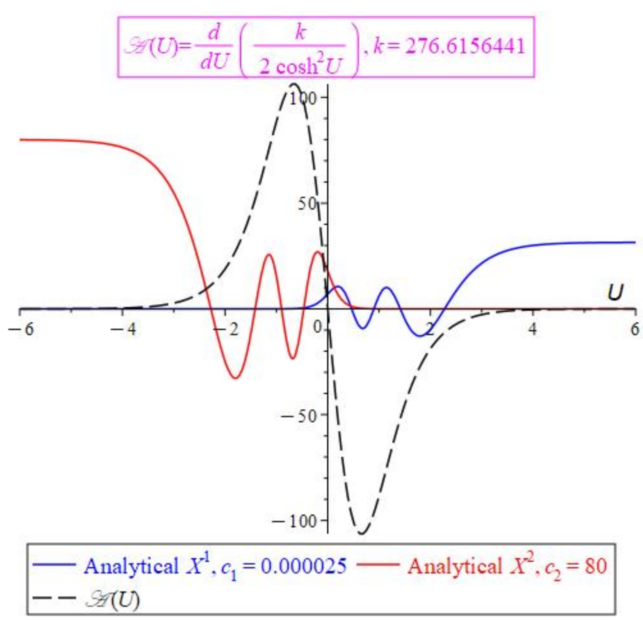


$\mathbf{m = 2}$

For dPT-flyby profile (17), both numerical and (semi-) analytical trajectories exhibit **DM** *for both components.*



$\mathbf{m = 3}$



$\mathbf{m = 4}$

Semi-analytic dPT trajectories for critical amplitude (i) $k_{crit} = 166\ldots.$ *& (ii)* $k_{crit} = 276\ldots.$ *in derivative-PT approx have half-wave numbers* $\mathbf{m = 3}$ *and* $\mathbf{m = 4}$*. (The red/blue curves are very small but do not vanish).*

Analytical solution for dPT*

Eqns (13a)-(13b) with dPT profile (17)

$$\frac{d^2}{dU^2}X^+(U) - \frac{1}{2}k\left(\tanh U - \tanh^3 U\right)X^+(U) = 0, \quad (19)$$

$$\frac{d^2}{dU^2}X^-(U) + \frac{1}{2}k\left(\tanh U - \tanh^3 U\right)X^-(U) = 0, \quad (20)$$

solved (computer software) $\rightsquigarrow$ confluent Heun fcts,

$$\begin{aligned} X^+(U) = {} & c_1 \mathrm{HeunC}\left(0,\,0,\,0,\,k,\,-\frac{k}{2},\,\frac{\tanh U + 1}{2}\right) \\ & + c_2 \mathrm{HeunC}\left(0,\,0,\,0,\,k,\,-\frac{k}{2},\,\frac{\tanh U + 1}{2}\right) \times \\ & \left(\int -\frac{1}{\mathrm{HeunC}^2\left(0,\,0,\,0,\,k,\,-\frac{k}{2},\,\frac{\tanh U+1}{2}\right)} dU\right), \end{aligned} \quad (21)$$

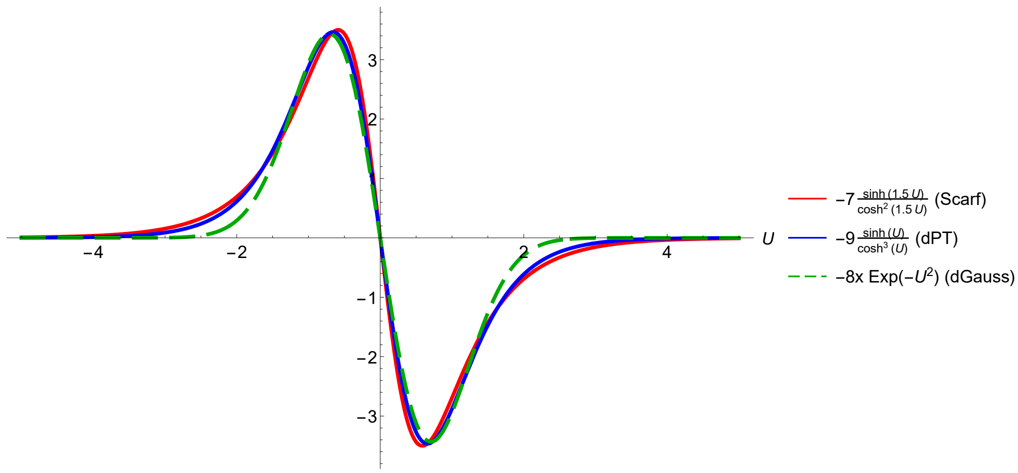
$$\begin{aligned} X^-(U) = {} & c_3 \mathrm{HeunC}\left(0,\,0,\,0,\,-k,\,\frac{k}{2},\,\frac{\tanh U + 1}{2}\right) \\ & + c_4 \mathrm{HeunC}\left(0,\,0,\,0,\,-k,\,\frac{k}{2},\,\frac{\tanh U + 1}{2}\right) \times \\ & \left(\int -\frac{1}{\mathrm{HeunC}^2\left(0,\,0,\,0,\,-k,\,\frac{k}{2},\,\frac{\tanh U+1}{2}\right)} dU\right) \end{aligned} \quad (22)$$

* Zhao: analytic sol. for dPT (unpublished)

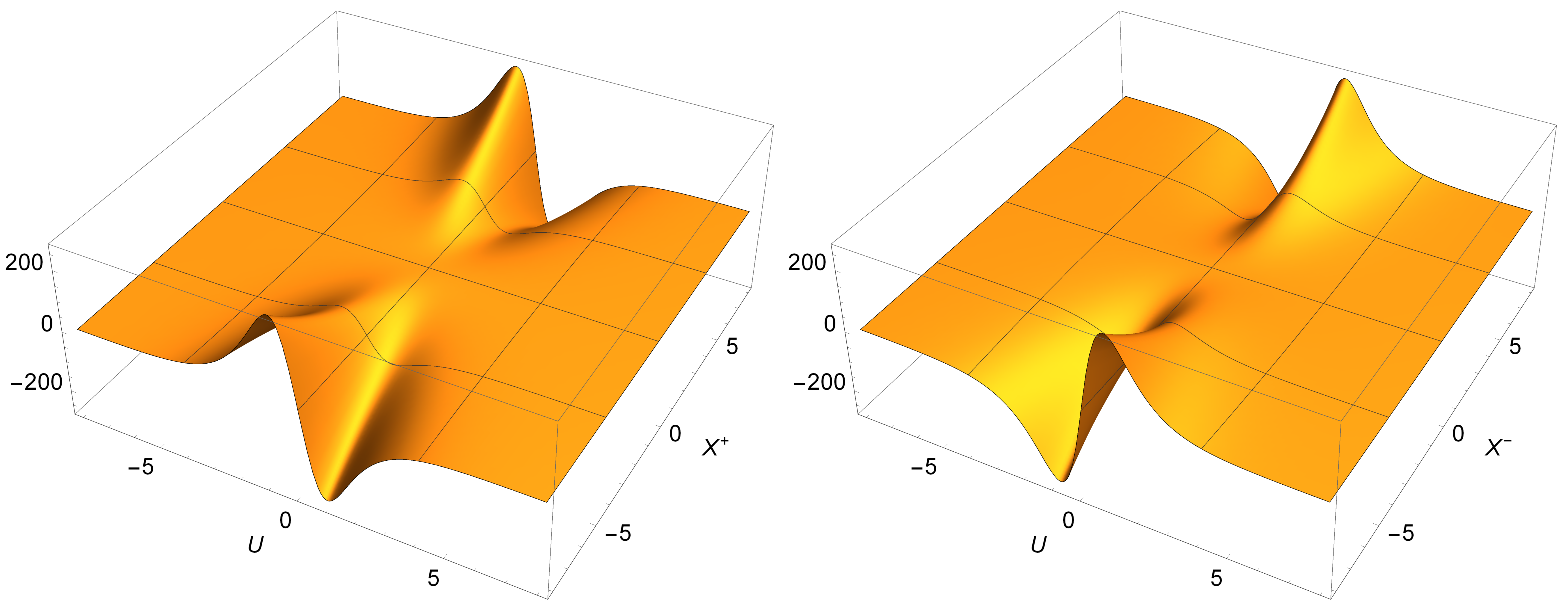
Scarf potential *

$$\mathcal{A}^{Scarf}(U) = 2g\frac{d}{dU}\left(\frac{1}{\cosh U}\right) = -2g\frac{\sinh U}{\cosh^{\mathbf{2}} U}, \quad (23)$$

cf. dPT (17).



$D = 2$ GW with Scarf wave profile†



*F. L. Scarf, "New Soluble Energy Band Problem", Phys. Rev. **112**, 1137-1140 (1958)

†D. E. Alvarez-Castillo and M. Kirchbach, "The real exact solutions to the hyperbolic Scarf potential," Rev. Mex. Fis. E **53**, 143-154 (2007)

Sols for $\mathcal{A}^{Scarf}$: $t = \sinh U$ [cf. $t = \tanh U$ for dPT]. Geo ens. (13a)-(13b) :

$$(1+t^2)\frac{d^2X^\pm}{dt^2} + t\frac{dX^\pm}{dt} \boxed{\mp g\frac{t}{1+t^2}} X^\pm = 0\,. \qquad (24)$$

Eqns uncoupled $\Rightarrow$ solved independently. Coefficient in front of linear-in-$X^\pm$ term whose sign change compensates $\mp$ sign-change, $t \to -t$, between components $\Rightarrow$ **same** g works for both components.

Silagadze: Eqn (24) is generalised hypergeometric type, solved analytically by Nikiforov-Uvarov algorithm* $\Rightarrow$

DM for quantized parameter

$$|g| = (2n+1)\sqrt{n(n+1)}\,, \quad n = 1, 2, \dots \qquad (25)$$

*P. M. Zhang, Z. K. Silagadze and P. A. Horvathy, "Flyby-induced displacement: analytic solution," Phys. Lett. **B** 868 (2015) 139687 [arXiv:2502.01326 [gr-qc]].

Trajectory

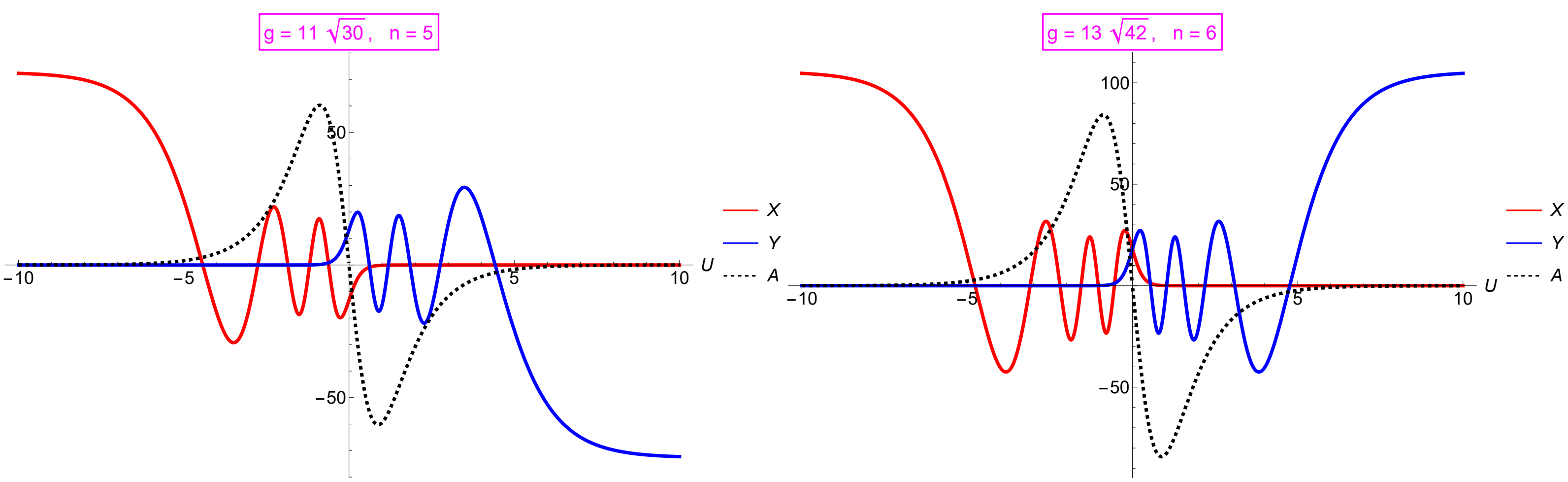
$$X_n(t) = (1+t^2)^{-\frac{n}{2}}\, e^{-\mathrm{sign}(g)\,\sqrt{n(n+1)}\,\arctan t}\, y_n(t)\,, \tag{26}$$

where $y_n(t) =$

$$\frac{B_n}{\rho(t)}\left[(1+t^2)^{-1/2} e^{-2\mathrm{sign}(g)\,\sqrt{n(n+1)}\,\arctan t}\right]^{(n)}\,.$$

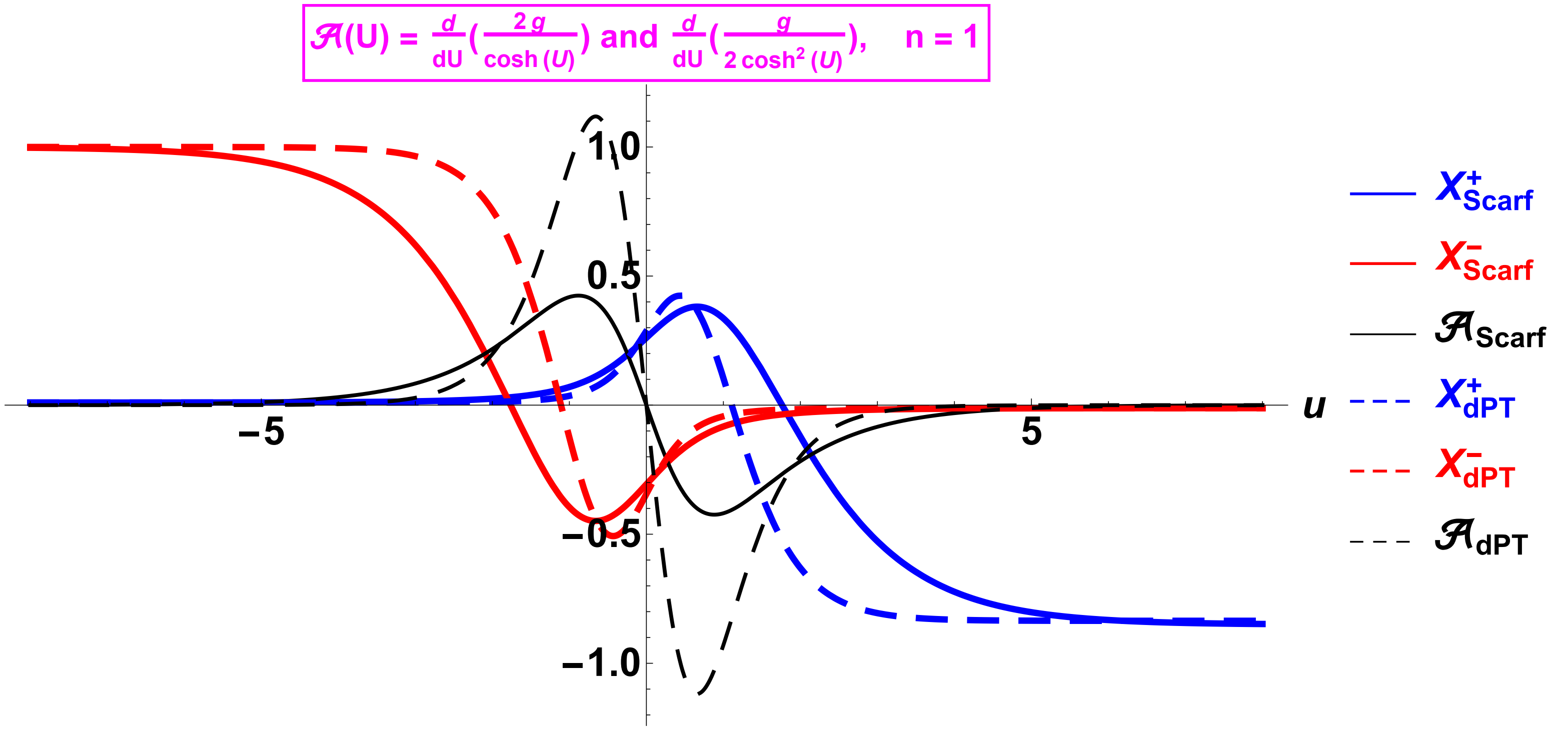
B_n const determined by initial conditions.

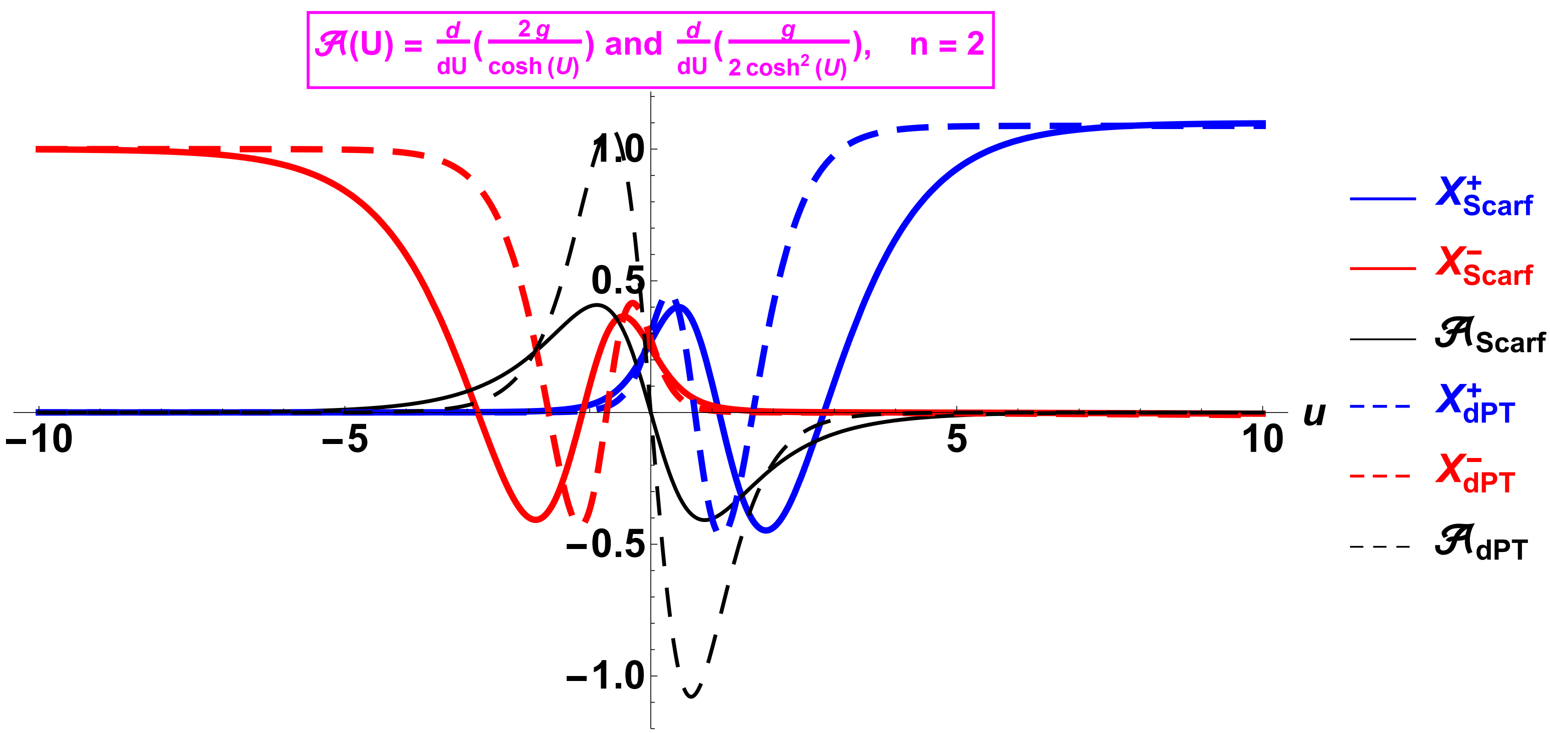
Precision also for high wave numbers



DM *geodesics for Scarf profile* (23) *for wave numbers* $\mathbf{n} = \mathbf{5}$ *and* $\mathbf{n} = \mathbf{6}$.

Scarf vs **dPT** (dashed)





Scarf vs dPT trajectories

Hodograph

$\equiv$ "Velocity diagram" in transverse space.

- $\mathbf{D = 1}$ transverse-space dim with coordinate Y,

$$W(U) = \frac{dY}{dU}\,. \tag{27}$$

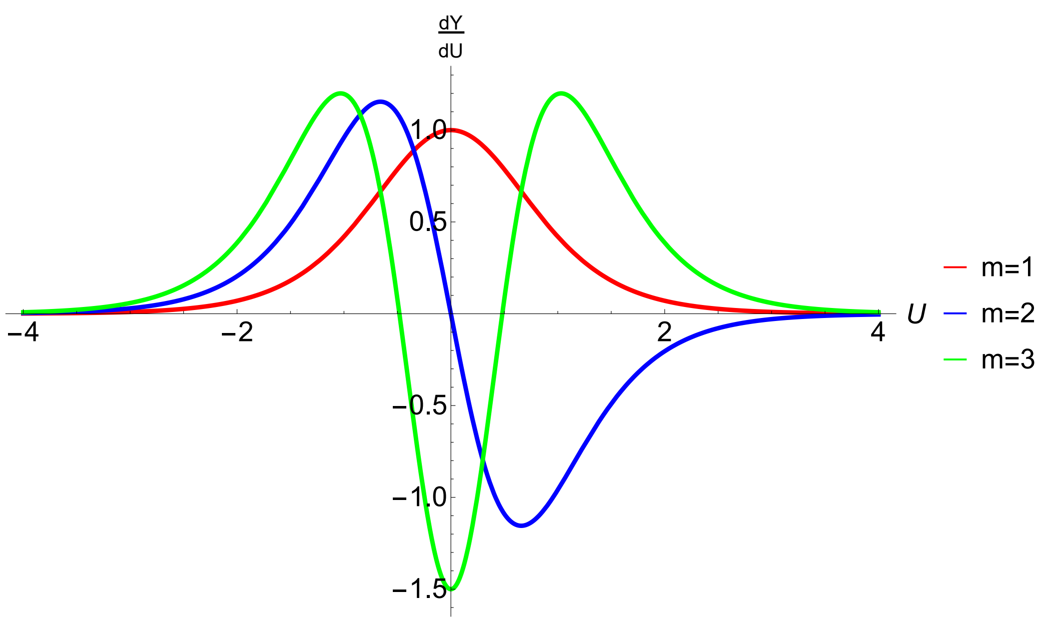
Particles in rest before sandwich wave arrives,

$$W(-\infty) = W_{-\infty} = 0\,. \tag{28}$$

Outgoing velocity constant by Newton's 1st law,

$$W(+\infty) = W_{\infty} = \text{const} \tag{29}$$

Generally: Velocity Effect, VM . **DM** , when $W_\infty \; = \; 0$

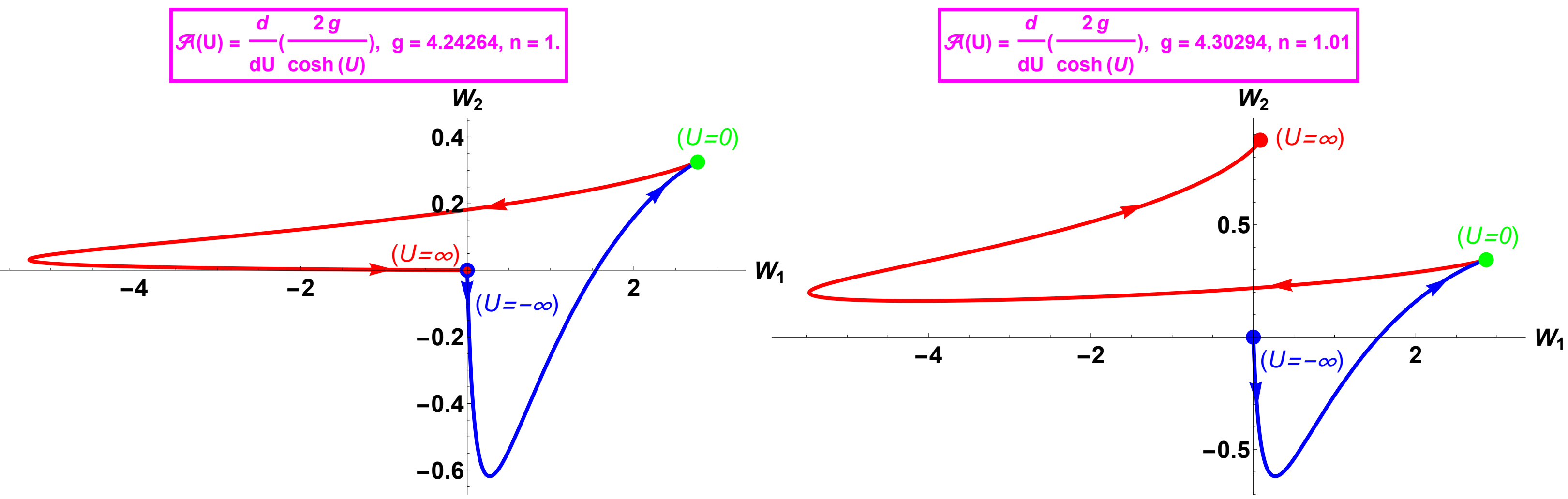


$\boxed{m \to n}$

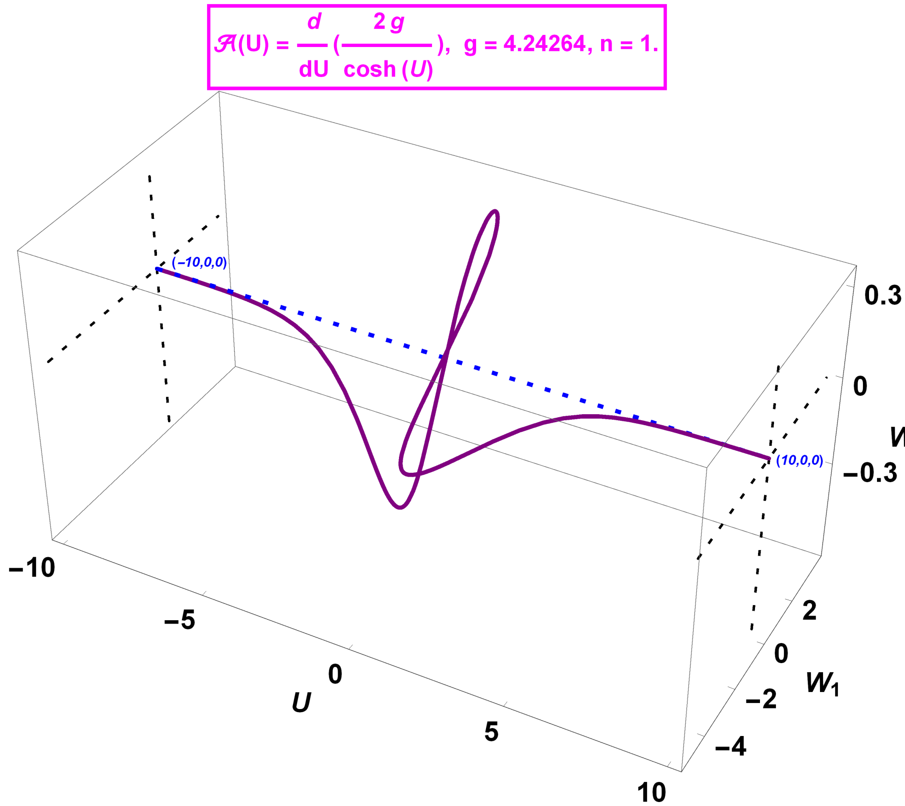
Unfolding hodograph : **DM** *for quantized Pöschl-Teller amplitude $k_n = \; 4n(n+1)$, $\sim \mathbf{n = 1, 2, 3}$ (half) waves.*

Similar behavior for dPT, Scarf.

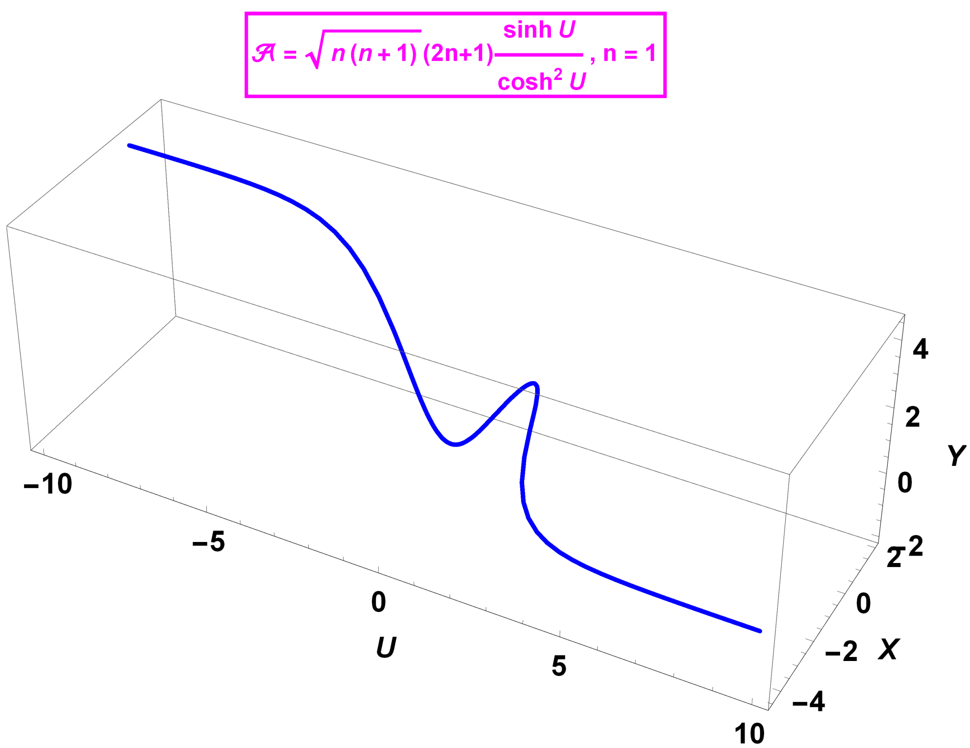
• $\mathbf{D = 2}$ transverse dim. For critical amplitude get closed curve. Shifting n to non-integer get VM, $\boldsymbol{W}(U = +\infty) \neq (0, 0)$. Unfolding to $(2+1)D$ yields curve which starts, and for **DM** also ends, at origin,



Scarf hodograph in $\mathbf{D = 2}$ transverse dimension. **DM** *with half wave number $\mathbf{n = 1}$ obtained for amplitude $g = g_1$ in (25). Velocity starts from $\boldsymbol{W}(U = -\infty) = (0, 0)$, follows* **blue** *curve to $\boldsymbol{W}(U = 0)$, then returns along the* **red** *curve to origin. When n is not integer, $\boldsymbol{W}(U = +\infty) \neq (0, 0)$, get VM.*



DM hodograph *unfolded to* $(2+1)D$*: curve starts and ends at* $\boldsymbol{W}(\pm\infty) = 0$.



DM trajectory *unfolded to* $(2+1)D$

CONCLUSION

VM : Particles at rest hit by a burst of GWs fly apart, moving freely along straight lines. **DM** is possible for exceptional values of wave parameters which correspond to having integer # of half-waves in wave zone.

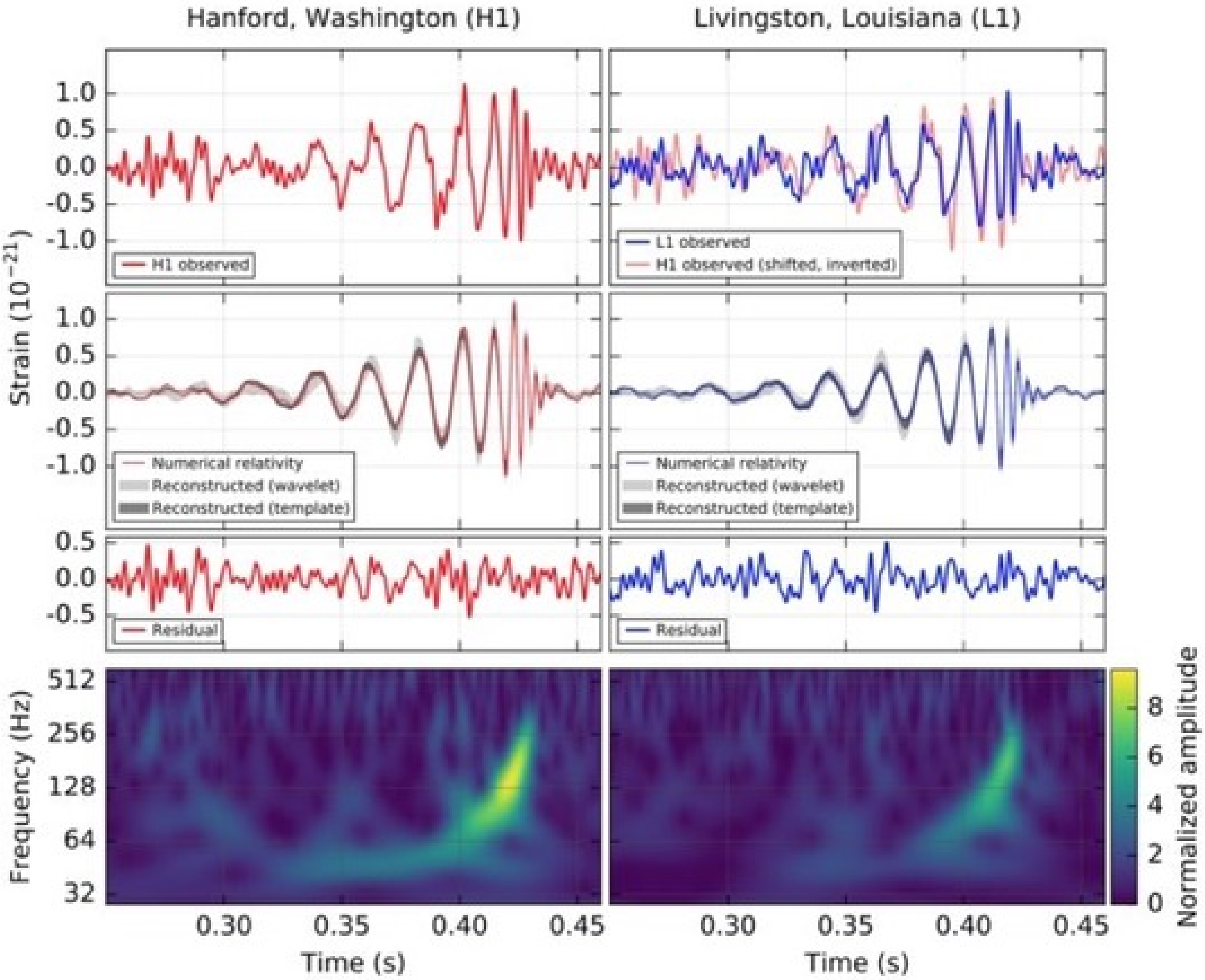
Clue: **DM** geodesics $\sim$ zero-energy sols of time-independent Sch eqn. (3a).

Analytic sols for PT, dPT, Scarf

TOY MODELS !

$\sim$ von Neumann's "dry water"

observed GW **GW170814**



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NO flyby-induced memory effect observed so far.